MATH 31B, LECTURE 1
MIDTERM 1
APRIL 20, 2012

Name: Solutions

Signature:

UID:

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Discussion meets: (circle one) Tuesday Thursday

Instructions: The exam is closed-book, closed-notes. Calculators are not permitted. Answer each question in the space provided. If the question is in several parts, carefully label the answer to each part. Do all of your work on the examination paper; scratch paper is not permitted. If you continue a problem on the back of the page, please write "continued on back".

Each problem is worth 10 points.

Problem	Score
1	
2	
3	
4	
5	
Total	

Problem 1: Evaluate the following limits.

(a) $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

(b) $\lim_{x\to 0^+} (\cos x)^{1/x^2}$.

a) Indeterminate $\frac{0}{0}$ form, use L'Hospital's rule.

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \overset{L.H}{=} \lim_{t\to 0} \frac{\frac{1}{2\sqrt{1+t}}+\frac{1}{2\sqrt{1-t}}}{1} = 1.$$

b) Indeterminate 1^{∞} form:

$$\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{x\to 0^+} e^{\ln\left[(\cos x)^{1/x^2}\right]}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2}}$$

Now

$$\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2} \overset{L.H.}{=} \lim_{x\to 0^+} \frac{\frac{-\sin x}{\cos x}}{2x} = -\frac{1}{2}\lim_{x\to 0^+} \frac{\sin x}{x} = -\frac{1}{2}$$

Problem 2:

(a) Evaluate the indefinite integral $\int \frac{\ln(\ln x)}{x \ln x}\, dx$.

(b) Use logarithmic differentiation to compute $f'(x)$, where $f(x) = \frac{\sqrt{x^3+2}\cdot(x-1)^{2/3}}{(4+x^2)^3}$

a) $u = \ln(\ln x)$, $du = \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$

$$\int \frac{\ln(\ln x)}{x\cdot \ln x}\,dx = \int u\,du = \frac{u^2}{2} + C$$

$$= \frac{(\ln(\ln x))^2}{2} + C.$$

b) $\ln f(x) = \frac{1}{2}\ln(x^3+2) + \frac{2}{3}\ln(x-1) - 3\ln(4+x^2)$

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{3x^2}{2(x^3+2)} + \frac{2}{3(x-1)} - \frac{6x}{4+x^2}$$

$$\Rightarrow f'(x) = \left(\frac{3x^2}{2(x^3+2)} + \frac{2}{3(x-1)} - \frac{6x}{4+x^2}\right)\left(\frac{\sqrt{x^3+2}\,(x-1)^{2/3}}{(4+x^2)^3}\right)$$

Problem 3: Let $f(x) = x^5 + 2x$

(a) Show that $f(x)$ is one-to-one on $(-\infty, \infty)$.

(b) Let $g(x) = f^{-1}(x)$. Find $g'(3)$.

[redacted]

(a) $f'(x) = 5x^4 + 2 \geq 2 > 0.$

So $f(x)$ is increasing, hence one-to-one.

b) $f(1) = 1^5 + 2 \cdot 1 = 3 \Rightarrow g(3) = 1$

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(1)} = \frac{1}{5 \cdot 1^4 + 2} = \frac{1}{7}.$$

Problem 4: A continuous annuity with withdrawal rate $N = \$1000/\text{year}$ and interest rate r is funded by an initial deposit of $P_0 = \$10000$. Let $P(t)$ be the balance of the account after t years.

(a) Write the differential equation satisfied by $P(t)$.

(b) What is the smallest value of r for which the annuity will never run out of money?

(c) If $r = 5\%$, at what time will the annuity run out of funds?

(a) $P'(t) = rP(t) - N = rP(t) - 1000$

(b) $P(t) = \frac{N}{r} + \left(P_0 - \frac{N}{r}\right)e^{rt}$

$= \frac{1000}{r} + \left(10000 - \frac{1000}{r}\right)e^{rt}$

We need $10000 - \frac{1000}{r} \geq 0$

$\Leftrightarrow 10 \geq \frac{1}{r} \Leftrightarrow r \geq .10 = \boxed{10\%}$

(c) $P(t) = \frac{N}{r} + \left(P_0 - \frac{N}{r}\right)e^{rt}$

$= \frac{1000}{.05} + \left(10000 - \frac{1000}{.05}\right)e^{.05t}$

$= 20000 - 10000\, e^{\frac{t}{20}}$

$P(t) = 0 \Leftrightarrow 20000 = 10000\, e^{\frac{t}{20}}$

$\ln 2 = t/20$

$\boxed{t = 20 \cdot \ln 2}$

Runs out after $20 \cdot \ln 2$ years.

Problem 5: The number of computers infected by a certain computer viruses increases at a rate proportional to the number of computers currently infected. Suppose that on January 20, 2012 there were 2^{10} computers infected with the virus, and three days later there were 2^{12} computers infected.

Let $P(t)$ be the number of computers infected t days after the virus was released (i.e. after the first computer was infected).

(a) What differential equation does $P(t)$ satisfy?

(b) Find the formula for $P(t)$.

(c) On what day was the virus released?

(a) $P'(t) = kP(t)$, where k is the growth rate.

(b) $P(t) = P_0 e^{kt} = e^{kt}$ since $P_0 = 1$.

Let T be the number of days between the release of the virus and January 20, 2012.

We have

$$P(T) = e^{kT} = 2^{10}$$

$$P(T+3) = e^{k(T+3)} = 2^{12}$$

So $$\frac{e^{k(T+3)}}{e^{kT}} = \frac{2^{12}}{2^{10}} = 2^2 = 4$$

$$e^{3k} = 4 \Rightarrow k = \frac{\ln 4}{3}$$

$$P(t) = e^{\frac{\ln 4}{3} \cdot t} = 4^{\frac{t}{3}}$$

c) We have

$$4^5 = 2^{10} = P(T) = 4^{\frac{T}{3}}$$

so $\frac{T}{3} = 5 \Rightarrow T = 15.$

January 20 is 15 days after the virus was released, so it was released on January 5, 2012